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CHARACTERIZATION THEOREMS INVOLVING THE GENERALIZED MARKOV-POLYA DAMAGE MODEL

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In the present paper, certain random damage models are examined, such as the Generalized Markov-Polya and the Quasi-Binomial, in which an interest in a random variable N is reduced to N_A. If N_B is the missing part, where $N = N_A + N_B$, the covariance between N_A and N_B is obtained for some general classes of distributions, such as the G.P.S.D. and M.P.S.D. for the random variable N. A characterization theorem is proved that under the generalized Markov-Polya damage model, the random variables N_A and N_B are the independent

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if and only if, N has the Generalized Polya-Eggenberger distribution. This generalizes the corresponding result for the Quasi-Binomial damage model and the generalized Poisson distribution. Finally, some interesting identities are obtained using the independence property and the covariance formulas between the numbers N_A and N_B.

ABSTRACT

In the present paper, certain random damage models are examined, such as the Generalized Markov-Polya and the Quasi-Binomial, in which an integer-valued random variable N is reduced to N_A. If N_B is the missing part, where $N = N_A + N_B$, the covariance between N_A and N_B is obtained for some general classes of distributions, such as the G.P.S.D. and M.P.S.D. for the random variable N. A characterization theorem is proved that under the generalized Markov-Polya damage model, the random variables N_A and N_B are independent if, and only if, N has the Generalized Polya-Eggenberger distribution. This generalizes the corresponding result for the Quasi-Binomial damage model and the generalized Poisson distribution. Finally, some interesting identities are obtained using the independence property and the covariance formulas between the numbers N_A and N_B.

Key words: damage model, Markov-Polya distribution, covariance identities, characterization

CHARACTERIZATION THEOREMS INVOLVING THE GENERALIZED MARKOV-POLYA DAMAGE MODEL

by

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1. INTRODUCTION

Let the integer-valued random variable N denote the size of a family, which produces two types of children, say boys and girls, for simplicity, with probabilities p and q, where $p + q = 1$. Let N_A and N_B denote the numbers of boys and girls, respectively, where $N = N_A + N_B$. If the accent is on the number N_A, we say that N is reduced to N_A by means of the binomial damage model

$$P(N_A = k|N = n) = \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, 2, \ldots. n.$$

It is well known that the numbers N_A and N_B are independent if, and only if, N has the Poisson distribution. Covariance formulas between N_A and N_B have been obtained by Raja Rao et al (1973) and Raja Rao (1981) for some general classes of distributions of N, such as the G.P.S.D. and the M.P.S.D. of Gupta (1974). These classes include many of the standard discrete distributions, generalized distributions, such as the Generalized Poisson, Negative Binomial and the Logarithmic Series distributions.

In the present paper, the Generalized Markov-Polya and the Quasi-Binomial random damage models are discussed. Covariance formulas for the numbers N_A and N_B are obtained for the G.P.S.D. and M.P.S.D. classes of distributions. A characterization theorem is proved for the Generalized Markov-Polya damage model, which says that N_A and N_B are independent if, and only if, N has the Generalized Polya-Eggenberger distribution. This result generalizes Consul's (1975) characterization of the Generalized Poisson distribution for the Quasi-Binomial damage model.

These characterization theorems together with the covariance formulas for the numbers N_A and N_B lead to some interesting identities. These identities involve the expectations of the sum of a random number of functions of the random variable N, where N has the Generalized Polya-Eggenberger or the Generalized Poisson distribution.

2. A GENERAL COVARIANCE FORMULA

Theorem (2.1): Suppose that an observation N is reduced to N_A according to some random damage model such that

$$E(N_A|N) = Np, \quad o < p < 1 \quad \ldots (2.1)$$

Further, let $N_A + N_B = N$. Then the covariance between the two random variables N_A and N_B is given by the equation

$$Cov(N_A,N_B) = p\,V(N) - V(N_A) \quad \ldots\ldots(2.2)$$

Proof: From equation (2.1) we get

$$E(N_A) = E\{E(N_A|N)\} = E(N)p$$

similarly $E(N_B) = E(N)q$, where $p + q = 1$. To find the variance of N_A, we use a result from Chiang (1968):

$$V(N_A) = V\{E(N_A|N)\} + E\{V(N_A|N)\}$$

$$= p^2 V(N) + E\{V(N_A|N)\} \quad \ldots.(2.3)$$

Similarly

$$Cov(N_A,N_B) = Cov\{E(N_A|N), E(N_B|N)\} + E\{Cov(N_A,N_B)|N\}$$

$$= pq\ V(N) + E\{Cov(N_A,N_B)|N\} \quad \ldots.(2.4)$$

Consider now

$$Cov(N_A,N_B)|N = E(N_A,N_B|N) - E(N_A|N)E(N_B|N)$$

$$= E\{N_A(N-N_A)|N\} - N^2pq$$

$$= N^2p - E(N_A^2|N) - N^2pq$$

$$= N^2p - \{V(N_A|N) + [E(N_A|N)]^2\} - N^2pq$$

$$= -V(N_A|N) \quad \ldots(2.5)$$

Using equation (2.5) in (2.4), we get

$$Cov(N_A,N_B) = pq\ V(N) - E\{V(N_A|N)\}.$$

An alternative formula is

$$Cov(N_A,N_B) = p\ V(N) - V(N_A),$$

which proves the theorem.

Observe that no special random damage model has been assumed, except that equation (2.1) holds, namely, $E(N_A|N) = Np$. This is a very general model. Further, the distribution of N is also left unspecified.

The following theorems are easily proved.

Theorem (2.2): Let the r.v. N have a Generalized Power Series Distribution (G.P.S.D.) with the series function f (θ), namely,

$$P(N=n) = a_n \frac{\theta^n}{f(\theta)}, \quad \theta > 0, \quad n \in T, \quad f(\theta) > 0, \quad a_n > 0$$

where T is a subset of the set of positive integers. Then, under the binomial damage model, (Raja Rao et al, 1973)

$$\mathrm{Cov}(N_A, N_B) = pq\theta^2 \frac{d^2}{d\theta^2} \log f \qquad \text{.....(2.5)}$$

It follows that

$$\mathrm{Cov}(N_A, N_B) \gtrless 0 \text{ according as } \log f(\theta) \text{ is Convex or Concave in } \theta.$$

This theorem includes many of the standard discrete distributions.

For Fisher's Logarithmic Series Distribution, we get

$$f(\theta) = -\log(1-\theta) \text{ and } \mathrm{Cov}(N_A, N_B) \gtrless 0 \quad \text{if } \theta \lesseqgtr 0.632.$$

Theorem (2.3): Let the r.v. N have a Modified Power Series Distribution (M.P.S.D.) with the probability function

$$P(N=n) = a_n \{g(\theta)\}^n / f(\theta), \quad a_n > 0, \quad g(\theta) > 0, \quad f(\theta) > 0, \quad n \in T.$$

Let the damage model be binomial, as before. Then (Raja Rao, 1981)

$$\mathrm{Cov}(N_A, N_B) = pq \{g(\theta)\}^2 \frac{d^2}{dg^2(\theta)} \log f(\theta) \qquad \text{.....(2.6)}$$

The M.P.S.D. class includes the Lagrangian (or Generalized) Poisson distribution, the Generalized Negative Binomial distribution and the Generalized Logarithmic Series Distribution , and their truncated forms, among others.

Theorem (2.3) shows that if $g(\theta)$ is an increasing function of θ, then N_A and N_B are positively or negatively correlated if the function $\log f(\theta)$ is convex or concave with respect to $g(\theta)$.

In the next section, we introduce the Generalized Markov-Polya damage model and obtain covariance formulas for N_A and N_B.

3. THE GENERALIZED MARKOV-POLYA DAMAGE MODEL AND ITS SPECIAL CASES

Definition(3.1): A r.v. N is reduced to N_A by the Generalized Markov-Polya Damage Model if the conditional distribution of N_A given $N=n$ is given by

$$P(N_A=x|N=n) = \binom{n}{x} \frac{ab}{a+b} \frac{(a+xt)^{(x,c)}}{a+xt} \frac{(b+\overline{n-x}t)^{(n-x,c)}}{(b+\overline{n-x}t)} \frac{(a+b+nt)}{(a+b+nt)^{(n,c)}} \ldots(3.1)$$

if $a > 0$, $b > 0$, $0 \leq t < 1$, $c \neq 0$, $x = 0,1,2,\ldots n$, (Janardan), 1977). Here

$$a^{(x,c)} = a\,(a+c)\,(a+2c)\,(a+3c)\,\ldots\,(a+\overline{x-1}\,c).$$

For example, $a^{(0,1)}=1$, $a^{(x,0)}=a^x$,

$$a^{(x,-1)} = a^{(x)} = a(a-1)(a-2)\ \ldots.\ (a-x+1)$$

$$a^{(x,1)} = a^{[x]} = a(a+1)\,(a+2)\ \ldots.\ (a+x-1)$$

$$(a+b)^{(x,c)} = a^x(1+\tfrac{b}{a})^{(x,c/a)} \quad \text{etc.}$$

A convenient and simple form of equation (3.1) is obtained by letting

$$p = a/(a+b), \quad q = b/(a+b), \quad \theta = t/(a+b), \quad \phi = c/(a+b)$$

Then the Generalized Markov-Polya damage model is

$$P(N_A = x|N=n) = \binom{n}{x} pq \frac{(p+x\theta)^{(x,\phi)}}{p+x\theta} \frac{(q+\overline{n-x}\theta)^{(n-x,\phi)}}{(q+\overline{n-x}\theta)} \frac{(1+n\theta)}{(1+n\theta)^{(n,\phi)}}, \quad \ldots \quad (3.2)$$

where $0<p<1$, $0<q<1$, $0\le\theta<1$, $\phi\neq 0$, $p+q=1$.

This model contains several distributions as special cases. For instance,

(i) $\theta=0$ and $\phi=0$ gives the binomial.

(ii) $\phi=0$ gives

$$P(N_A = x|N=n) = \binom{n}{x} \frac{B_x(p,\theta)\ B_{n-x}(q,\theta)}{B_n(1,\theta)}$$

where $B_x(p,\theta) = p(p+x\theta)^{x-1}$. This is the Quasi-Binomial distribution.

Because of its importance in the sequel, we define this distribution as follows:

<u>Definition</u> (3.2) A discrete r.v. N is reduced to N_A by the Quasi-Binomial damage model if the Conditional distribution of N_A given N=n is given by

$$P(N_A=x|N=n) = \binom{n}{x} \frac{pq}{1+n\theta} \left(\frac{p+x\theta}{1+n\theta}\right)^{x-1} \left(\frac{q+\overline{n-x}\theta}{1+n\theta}\right)^{n-x-1} \quad \ldots. \quad (3.3)$$

where $p+q = 1$, $p \geq 0$, $\theta < 1$ and $x = 0,1,2,\ldots,n$. This reduces to the binomial damage model if $\theta=0$.

(iii) $\theta=0$ gives the Markov-Polya distribution,

$$P(N_A=x|N=n) = \binom{n}{x} \frac{\prod_{j=0}^{x-1} (p+j\phi) \prod_{j=0}^{n-x-1} (q+j\phi)}{\prod_{j=0}^{n-1} (1+j\phi)}$$

(iv) $\phi=-1$ gives the Quasi-Hypergeometric distribution,

$$P(N_A=x \mid N=n) = \frac{H_x(p,\theta)\, H_{n-x}(q,\theta)}{H_n(1,\theta)}$$

where

$$H_x(p,\theta) = \frac{p}{p + x\theta} \binom{p + x\theta}{x}.$$

(v) $\phi=-1$ and $\theta=0$ gives the hypergeometric distribution.

(vi) $\phi=+1$ gives the Quasi-negative hypergeometric distribution.

(vii) $\theta=1$, $\phi=-1$ or $\theta=0$, $\phi=1$ gives the negative hypergeometric or (the beta-binomial) distribution.

4. THE GENERALIZED POLYA-EGGENBERGER DISTRIBUTION AND ITS SPECIAL CASES

<u>Definition</u> (4.1): A random variable N is said to have the Generalized Polya-Eggenberger distribution, if its probability function is given by (Janardan, 1973)

$$P(N=n) = \frac{p}{p+n\theta} \frac{(p+n\theta)^{(n,\phi)}}{n!} \frac{\beta^n (1-\beta)^{\frac{p+n\theta}{\phi}}}{\phi^n} \quad \ldots \, (4.1)$$

when $0\leq\theta<1$, $\phi\neq0$, $n=0,1,2,\ldots$.

Some special cases of this distribution are the following:

(i) $\theta=0$ gives the Polya-Eggenberger distribution (i.e., the negative binomial distribution with $p=1-\beta$ and $\gamma=p/\phi$.)

(ii) $\phi=1$ gives the generalized negative binomial distribution

$$P(N=n) = \frac{p}{p+n\theta} \quad \frac{(p+n\theta)^{(n,1)} \quad \beta^n (1-\beta)^{p+n\theta}}{n!}, \quad 0\leq\beta<1.$$

$$= \frac{p}{n!} \quad \frac{\Gamma(p+n(\theta+1))}{\Gamma(p+n\theta+1)} \quad \frac{[\beta(1-\beta)^\theta]^n}{(1-\beta)^{-p}}$$

where $g(\beta) = \beta(1-\beta)^\theta$, $f(\beta) = (1-\beta)^{-p}$.

(iii) In the Generalized Polya-Eggenberger distribution, if we take $\frac{\theta}{p} = \lambda$ and let $\phi \to 0$ such that $\frac{p\beta}{\phi} \to M$, it can be shown that the resulting distribution is the Generalized Poisson distribution. We define this distribution for easy reference.

Definition (4.2): A discrete r.v. N is said to have the Generalized Poisson (or Lagrangian Poisson) distribution if its probability function is

$$P(N=x) = M^x (1+x\lambda)^{x-1} \frac{e^{-M(1+x\lambda)}}{x!}, \quad x=0,1,2....... \qquad ...(4.2)$$

where $M>0$, $0\leq\lambda<M^{-1}$.

The Generalized Poisson distribution is also a limiting form of the Quasi-Binomial distribution if p and θ are very small while n is large such that np and nθ are constant.

5. A CHARACTERIZATION THEOREM

Theorem (5.1): Let a r.v. N be reduced to N_A by means of the Generalized Markov-Polya random damage model $S(k|n) = P(N_A = k|N=n)$ given by equation (3.2). Let $N_B = N - N_A$. Then the random variables N_A and N_B are independent if, and only if, the r.v. N has the Generalized Polya-Eggenberger distribution.

<u>Proof:</u> Necessity follows easily since the damage model is Generalized Markov-Polya, we get the conditional probability

$$P(N_A=k,\ N_B=n-k \mid N=n) = \binom{n}{k}\, pq\ \frac{(p+k\theta)^{(k,\phi)}}{(p+k\theta)}\ \frac{(q+\overline{n-k}\theta)^{(n-k,\phi)}}{(q+\overline{n-k}\theta)}\ \frac{(1+n\theta)}{(1+n\theta)^{(n,\phi)}} \quad \ldots (5.1)$$

This gives the unconditional probability

$$P(N_A=k,\ N_B=n-k) = P(N_A=k,\ N_B=n-k \mid N=n)\cdot P(N=n)$$

If N has the Generalized Polya-Eggenberger distribution with parameters $(1, \theta, \phi, \beta)$ it is clear that $P(N_A=k,\ N_B=n-k)$ is factorizable, showing that N_A and N_B are independent.

To prove sufficiency, let N_A and N_B be independent. Denote $\Pi_n = P(N=n)$. Then following Kruskal's (1960) approach, we have

$$\Pi_{u+v}\ \frac{(1+\overline{u+v}\theta)}{(1+\overline{u+v}\theta)}\ \frac{(u+v)!}{(u+v,\phi)} = f(u)\cdot g(v) \quad \ldots (5.2)$$

for some functions $f(\cdot)$ and $g(\cdot)$. Neither $f(0)$ nor $g(0)$ can be zero, for there is a positive probability that $N_A=0$ and that $N_B=0$. Thus there is a function $h(\cdot)$ such that

$$f(u)\ g(v) = h(u+v) \quad \ldots (5.3)$$

for some non-negative integers. This is the Cauchy functional equation, whose non-trivial solution is

$$f(u) = \alpha\, e^{\lambda u}, \quad g(v) = \alpha'\, e^{\lambda v}$$

so that from eq (5.3)

$$\Pi_{u+v} \frac{(1+\overline{u+v}\ \theta)\ (u+v)!}{(1+\overline{u+v}\ \theta)^{(u+v,\phi)}} = \alpha\, \alpha' e^{\lambda(u+v)}$$

or

$$\Pi_n = \frac{1}{(1+n\theta)} \quad \frac{(1+n\theta)^{(n,\phi)}}{n!} \quad \alpha\, \alpha' e^{\lambda n} .$$

Setting $e^{\lambda} = \frac{\beta(1-\beta)^{\theta/\phi}}{\phi}$ and using the fact that $\Sigma \Pi_n = 1$, we get $\alpha\, \alpha' = (1-\beta)^{1/\phi}$. Therefore N has the Generalized Polya-Eggenberger distribution with parameters $(1, \theta, \phi, \beta)$.

Remark: It is seen from Theorem (5.1) that the numbers N_A and N_B have independent Generalized Polya-Eggenberger distributions with probability functions

$$P(N_A=x) = \frac{p}{p+x\theta} \quad \frac{(p+x\theta)^{(x,\phi)}}{x!} \quad \beta^x \quad \frac{(1-\beta)^{\frac{p+x\theta}{\phi}}}{\phi^x}$$

and

$$P(N_B = y) = \frac{q}{q+y\theta} \quad \frac{(q+y\theta)^{(y,\phi)}}{y!} \quad \frac{\beta^y (1-\beta)^{\frac{q+y\theta}{\phi}}}{\phi^y} .$$

6. COVARIANCE BETWEEN THE NUMBERS N_A and N_B

Theorem (6.1): Let the r.v. N have any discrete distribution. Further suppose that the r.v. N is reduced to N_A by the Generalized Markov-Polya damage model, given by (3.1). If $N_A + N_B = N$, the covariance between N_A and N_B is given by equation (6.5).

Proof: Observe that in the Generalized Markov-Polya damage model, we do have $E(N_A|N) = N\frac{a}{a+b} = Np$, so that Theorem (2.1) applies. This gives

$$\text{Cov}(N_A,N_B) = pV(N) - V(N_A) = pqV(N) - E\{V(N_A|N)\} \quad . . . (5.1)$$

Also from Janardan and Schaeffer (1977) we know that

$$V(N_A|N) = \frac{ab}{a+b}\left[\frac{N^2}{a+b} - \sum_{j=0}^{N-1} \frac{N^{(j+2)}\,(t+c)^j}{(a+b+Nt+\overline{N-j+1}\ c)^{(j+1,c)}}\right]$$

$$= N^2pq - pq\sum_{j=0}^{N-1} \frac{(a+b)\ N^{(j+2)}\,(t+c)^j}{(a+b+Nt+\overline{N-j+1}\ c)^{(j+1,c)}} \quad . . . (6.2)$$

Defining $\theta = \frac{t}{a+b}$ and $\phi = \frac{c}{a+b}$, the denominator of the jth term may be written as

$$(a+b+Nt+\overline{N-j+1}\ c)^{(j+1,c)} = (a+b)^{j+1}\ (1+N\theta+\overline{N-j+1}\ \phi)^{(j+1,\phi)} \quad . . . (6.3)$$

This gives from equation (6.2)

$$V(N_A|N) = N^2\,pq - pq\sum_{j=0}^{N-1} \frac{N^{(j+2)}\,(\theta+\phi)^j}{(1+N\theta+\overline{N-j+1}\ \phi)^{(j+1,\phi)}} \quad . . . (6.4)$$

Substituting eq (6.4) in eq (6.1), we obtain

$$\text{Cov}(N_A,N_B) = pq\ \{V(N) - E(N^2) + E\sum_{j=0}^{N-1} \frac{N^{(j+2)}\,(\theta+\phi)^j}{(1+N\theta+\overline{N-j+1}\ \phi)^{(j+1,\phi)}}\}$$

i.e.,

$$\text{Cov}(N_A,N_B) = pq\ \{E\sum_{j=0}^{N-1} \frac{N^{(j+2)}\,(\theta+\phi)^j}{(1+N\theta+\overline{N-j+1}\ \phi)^{(j+1,\phi)}} - E^2(N)\} \quad . . . (6.5)$$

Remarks: (1) An important special case occurs when we take $\theta=\phi=0$, which is equivalent to taking $t=c=0$ in the Generalized Markov-Polya damage model, which reduces to the binomial damage model and gives

$$\text{Cov}(N_A, N_B) = pq\ \{V(N) - E(N)\},$$

as in Raja Rao et al (1973).

2) Another important case is when the r.v. N has the Modified Power Series distribution with p.f. as in Theorem (2.3). Since

$$E(N) = \frac{f'(\theta)\ g(\theta)}{g'(\theta)\ f(\theta)},$$

eq (6.5) gives

$$\text{Cov}(N_A, N_B) = pq\ \{E \sum_{j=0}^{N-1} \frac{N^{(j+2)}\ (\theta+\phi)^j}{(1+N\theta+\overline{N-j+1}\ \phi)^{(j+1,\phi)}} - \left[\frac{f'(\theta)\ g(\theta)}{g'(\theta)\ f(\theta)}\right]^2\}. \quad . \quad . \quad (6.6)$$

where the expectation is taken w.r.t. the M.P.S.D. The corresponding results for the Generalized Poisson, Generalized Negative Binomial and the Generalized Logarithmic Series Distributions are obtained by suitably choosing the functions $f(\theta)$ and $g(\theta)$.

For the Generalized Poisson, $f(M) = e^{M}$ and $g(M) = M\ e^{-\lambda M}$. We get

$$\text{Cov}(N_A, N_B) = pq\ \{E \sum_{j=0}^{N-1} \frac{N^{(j+2)}\ (\theta+\phi)^j}{(1+N\theta+\overline{N-j+1}\ \phi)^{(j+1,\phi)}} - \frac{M^2}{(1-\lambda M)^2}\} \quad . \quad . \quad . \quad (6.7)$$

For the Generalized Negative binomial, $f(\beta) = (1-\beta)^{-p}$ and $g(\beta) = \beta(1-\beta)^{\theta}$

This gives

$$\text{Cov}(N_A, N_B) = pq\{\ E \sum_{j=0}^{N-1} \frac{N^{(j+2)}\ (\theta+\phi)^j}{(1+N\theta+\overline{N-j+1}\ \phi)^{(j+1,\phi)}} - \frac{p^2\beta^2}{\{1-\beta(1+\theta)\}^2}\} \quad . \quad . \quad . \quad (6.8)$$

3). An interesting case occurs when the r.v. N has the Generalized Polya-Eggenberger distribution, as in eq (3.3). Since $E(N) = \frac{p\theta}{\phi(1-\theta)-\theta\beta}$, we get from equation (6.5),

$$Cov(N_A,N_B) = pq \left\{ E \sum_{j=0}^{N-1} \frac{N^{(j+2)}\,(\theta+\phi)^j}{(1+N\theta+\overline{N-j+1}\,\phi)^{(j+1,\phi)}} - \frac{p^2\theta^2}{\{\phi(1-\theta)-\theta\beta\}^2} \right\} \quad . . . (6.9)$$

But in Theorem (5.1) we have proved that N_A and N_B are independent. This gives an interesting identity, which we summarize in the form of a theorem.

<u>Theorem</u> (5.2): Let the r.v. N have the Generalized Polya- Eggenberger distribution given by eq (4.1). Then the following identity holds:

$$E \left\{ \sum_{j=0}^{N-1} \frac{N^{(j+2)}\,(\theta+\phi)^j}{(1+N\theta+\overline{N-j+1}\,\phi)^{(j+1,\phi)}} \right\} = \{E(N)\}^2 = \frac{p^2\theta^2}{\{\phi(1-\theta)-\theta\beta\}^2} \quad . . . (6.10)$$

Observe that in equation (6.10), one has on the left hand side the sum of a random number of functions of the r.v. N, and the expectation is to be taken w.r.t. the Generalized Polya-Eggenberger distribution.

<u>Remark</u> (1): As we have mentioned in Section 3, taking $\phi = 0$ in the Generalized Markov-Polya distribution gives the Quasi-Binomial damage model [definition (3.3)]. Using $\frac{\theta}{p} = \lambda$ and letting $\phi \to 0$ such that $\frac{p\theta}{\phi} \to M$, one obtains the Generalized Poisson distribution from the Generalized Polya-Eggenberger distribution. Making these parametric limiting operations, Theorem (5.1) reduces to a characterization theorem concerning the Generalized Poisson distribution with the Quasi-Binomial damage model. In this sense, our Theorem (5.1) generalizes Consul's (1975) result.

<u>Remark</u> (2): If we let c=0, i.e. $\phi = 0$, in equation (6.5), we get the result:

$$Cov(N_A,N_B) = pq \left\{ E \sum_{j=0}^{N-1} \frac{N^{(j+2)}\,\theta^j}{(1+N\theta)^{(j+1)}} - E^2(N) \right\} \quad . . . (6.11)$$

for the Quasi-Binomial damage model, whatever be the distribution of N.

Similarly taking $\phi=0$ in equation (6.6) gives $Cov(N_A, N_B)$, whenever N has the M.P.S.D.

Choosing, in particular, N to have the Generalized Poisson distribution gives, from equation (6.10), the identity

$$E\left\{\sum_{j=0}^{N-1} \frac{N^{(j+2)}\,\theta^{j}}{(1+N\theta)^{(j+1)}}\right\} = \frac{M^{2}}{(1-M\lambda)^{2}} \quad \ldots \quad (6.12)$$

<u>Remark</u> (3): Making certain other parametric limiting operations, it is possible to obtain a series of characterization theorems as special cases from our Theorem (5.1). Some examples are as follows:

Choosing $\phi=1$, one obtains the quasi-negative hypergeometric distribution from the Generalized Markov-Polya distribution. Similarly choosing $\phi=1$, we get the Generalized Negative Binomial distribution. The result may be stated as

<u>CORROLARY (1) to Theorem (5.1)</u>: Let a r.v. N be reduced to N_A by means of the quasi-negative hypergeometric damage model. Let $N_B = N-N_A$. Then the random variables N_A and N_B are independent if, and only if, N has the Generalized Negative Binomial distribution.

<u>CORROLARY (2)</u>: Let a r.v. N be reduced to the number N_A according to the negative hypergeometric (the beta-binomial) damage model. Then the numbers N_A and N_B are independent if, and only if, N has the negative Binomial distribution.

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